Exploring a Structure for Mathematics Lessons that Foster Problem Solving and Reasoning

Peter Sullivan
Monash University
<peter.sullivan@monash.edu>

Nadia Walker
Monash University
<Nadia.walker@monash.edu>

Chris Borcek
St Kilda Primary School
<borcek.christine.c@edumail.vic.gov.au>

Mick Rennie
St Kilda Primary School
<rennie.michael.a1@edumail.vic.gov.au>

While there is widespread agreement on the importance of incorporating problem solving and reasoning into mathematics classrooms, there is limited specific advice on how this can best happen. This is a report of an aspect of a project that is examining the opportunities and constraints in initiating learning by posing challenging mathematics tasks intended to prompt problem solving and reasoning to students, not only to activate their thinking but also to develop an orientation to persistence. The results indicate that such learning is facilitated by a particular lesson structure. This article reports research on the implementation of this lesson structure and also on the finding that students' responses to the lessons can be used to inform subsequent learning experiences.

Introduction

Teachers commonly report experiencing difficulties in incorporating problem solving and reasoning into their mathematics classrooms while at the same time catering for students with a wide range of prior experiences. Rather than the common approach of starting learning sequences with simple tasks intending to move to more challenging tasks subsequently, we are exploring an approach based on initiating learning through a challenging task — described as activating cognition. In particular, we describe the implementation of a particular lesson structure designed to initiate learning through an appropriate challenge, effectively differentiating that challenge for particular student needs, and consolidating the learning through task variations.

The data reported below are from one aspect of a larger project[1] that is exploring the proposition that students learn mathematics best when they engage in building connections between mathematical ideas for themselves (prior to instruction from the teacher) at the start of a sequence of learning rather than at the end. The larger project is studying the type of tasks that can be used to prompt this learning and ways that those tasks can be optimally used, one aspect of which is communicating to students that this type of learning requires persistence on their part. Essentially the notion is for teachers to present tasks that the students do not yet know how to answer and to support them in coming to find a solution for themselves.

There are many scholars who have argued that the choice of task is fundamental to opportunities for student problem solving and reasoning. Anthony and Walshaw (2009), for example, in a research synthesis, concluded that "in the mathematics classroom, it is through tasks, more than in any other way, that opportunities to learn are made available to

[1] The Encouraging Persistence Maintaining Challenge project was funded through an Australian Research Council Discovery Project (DP110101027) and was a collaboration between the Monash University and Australian Catholic University. The views expressed are those of the authors. The generous participation of project schools is acknowledged.

2015. In M. Marshman, V. Geiger, & A. Bennison (Eds.). *Mathematics education in the margins* (Proceedings of the 38th annual conference of the Mathematics Education Research Group of Australasia), pp. 41–56. Sunshine Coast: MERGA.

the students" (p. 96). Similar comments have been made by Ruthven, Laborde, Leach, and Tiberghien (2009) and Sullivan, Clarke, and Clarke (2013).

There are also scholars who have proposed that those tasks should be appropriately challenging. Christiansen and Walther (1986), for example, argued that non-routine tasks, because they build connections between different aspects of learning, provide optimal conditions for thinking in which new knowledge is constructed and earlier knowledge is activated. Similarly, Kilpatrick, Swafford, and Findell (2001) suggested that teachers who seek to engage students in developing adaptive reasoning and strategic competence, or problem solving, should provide them with tasks that are designed to foster those actions. Such tasks clearly need to be challenging and the solutions needs to be developed by the learners. This notion of appropriate challenge also aligns with the *Zone of Proximal Development* (ZPD) (Vygotsky, 1978). Similarly, the National Council of Teachers of Mathematics (NCTM) (2014) noted:

> Student learning is greatest in classrooms where the tasks consistently encourage high-level student thinking and reasoning and least in classrooms where the tasks are routinely procedural in nature. (p. 17)

This approach was described in *PISA in Focus* (Organisation for Economic Co-operation and Development (OECD) (2014) as follows:

> Teachers' use of cognitive-activation strategies, such as giving students problems that require them to think for an extended time, presenting problems for which there is no immediately obvious way of arriving at a solution, and helping students to learn from their mistakes, is associated with students' drive. (p. 1)

The OECD (2014) explicitly connected student drive, which we associate with persistence, with higher achievement.

There are many research findings that elaborate how such advice can be implemented in classrooms, some of which is reviewed below. This report seeks to extend this advice in three significant ways: first, by investigating a specific lesson structure and particular tasks; second, by suggesting how such tasks can be adapted to accommodate differences in students' prior experiences; and third, by considering how the learning from the challenging tasks can be consolidated.

The Connection Between the Research Framework and the Structuring of Lessons

The data reported below are informed by a framework as shown in Figure 1, adapted from Clark and Peterson (1986), that proposes that teachers' intentions to act are informed by their knowledge, their disposition, and the constraints they anticipate experiencing. The particular focus in this article is the ways that each of these factors connect to the structuring of lessons.

One node of this framework presents decisions on lesson structure as being informed by the knowledge of the teacher. The different aspects of such knowledge, specifically teachers' knowledge of mathematics, of pedagogy and of students, are represented schematically by Hill, Ball, and Schilling (2008).

The inference is that it is more likely that teachers will intend to use challenging tasks if they understand the mathematics and its potential, are aware of approaches to implementing the tasks in classrooms and can anticipate student responses.

Another node in the framework suggests that teachers' planning intentions are informed by their dispositions including their beliefs about how students learn (Zan, Brown, Evans, & Hannula, 2006), the ways that challenge can activate cognition (Middleton, 1995), and perspectives on self-goals, a growth mindset and the importance of student persistence (Dweck, 2000).

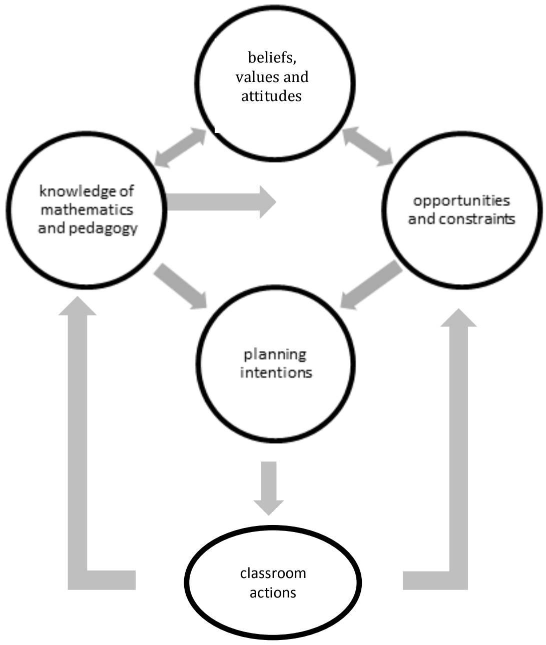

Figure 1: The framework informing the research.

A third node proposes that the ways teachers plan are influenced by constraints that they anticipate they might experience. For example, teachers may be more likely to enact lessons based on challenging tasks if they do not fear negative reactions from students (see Desforges & Cockburn, 1987).

These three nodes interact with each other and together they inform teachers' planning intentions which in turn influence the classroom actions.

An analogous framework that similarly connects teachers' knowledge, beliefs and perceptions was presented in a diagram by Stein, Grover, and Henningsen (1996), which, when converted to text, proposes that the features of the mathematical task when set up in the classroom, as well as the cognitive demands it makes of students, are informed by the mathematical task. These features are, in turn, influenced by the teacher's goals, subject-matter knowledge, and their knowledge of their students. This then informs the mathematical task as experienced by students which creates the potential for their learning.

The particular lesson structure being explored addresses four aspects arising from consideration of both frameworks, specifically:

- The ways the tasks are posed in the introductory phase that is described by Lappan, Fey, Fitzgerald, Friel, and Phillips (2006) as the Launch, and by Inoue (2010), in outlining the structure of Japanese lessons as Hatsumon, meaning the initial problem, and Kizuki which is what it is intended that the students will learn;
- Actions taken to differentiate the task for particular students that occur during what Lappan et al. call Explore, and what Inoue describes as Kikanjyuski which is the individual or group work on the problem. Note that Inoue uses the term Kikan shido to suggest that the teacher actions during this aspect include thoughtfully walking around the desks;
- Ways that the student activity on the task is reviewed, described by Lappan et al. as Summary; and which includes both what Inoue calls Neriage which is carefully

managed whole class discussion seeking the students' insights, and Matome which is the teacher summary of the key ideas; and
- Subsequent teacher actions to pose additional experiences that consolidate the learning activated by engaging with the initial task.

The four aspects are elaborated below. In each of the aspects, teachers' actions connect directly to their knowledge of the mathematics involved in the task, their beliefs about what students can do and their anticipation of any constraints they may experience.

Posing the Task

A key aspect of the structuring of a lesson is the information provided to students as part of the introduction. If the teacher is working on the proposition that the students can be offered the opportunity to explore the problem and associated mathematics for themselves, then the introductory phase of the lesson becomes critical. Jackson, Garrison, Wilson, Gibbons, and Shahan (2013), for example, argued that there are two key issues for teachers to consider in the *set up* of the task. The first is that a common language can be established not only for students to interpret the task appropriately but also so they can contribute to the subsequent discussion. Second, it is productive if teachers consciously maintain the cognitive demand of the task. It can be assumed that decisions teachers make in maintaining the challenge are directly connected to their knowledge and beliefs about mathematics and pedagogy. Also connected to the maintenance of the challenge is whether teachers anticipate negative student reactions. Interestingly, in an earlier iteration of the project, Sullivan, Askew, Cheeseman, Clarke, Mornane, Roche, and Walker (2014) found that the majority of students do not fear challenges: they welcome them. Further, rather than preferring teachers to instruct them on solution methods, many students reported that they prefer to work out solutions and representations for themselves or with a partner.

Differentiating the Task

A second aspect of structuring lessons is anticipating ways that different students within the class might respond to the challenge, noting that this is important whether the students are grouped by their achievement or not. Sullivan, Mousley, and Jorgensen (2009) described two key actions in differentiating the experience:

- The provision of enabling prompts, which involve reducing the number of steps, simplifying the complexity of the numbers, and varying the forms of representation for those students who cannot proceed with the task with the explicit intention that they work on the initial task subsequently; and
- Offering extending prompts to students who complete the original task quickly which ideally elicit abstraction and generalisation of the solutions.

These differentiated experiences are offered after students have engaged with the original task for some time and have the same characteristics as the original task, meaning that students engage with the task for themselves as distinct from being told what to do.

Reviewing Student Activity on the Task

A further key aspect of the structuring of lessons is the review of students' solutions and strategies on the challenging task. The key elements of such lesson reviews were described by Smith and Stein (2011) as:

- Selecting particular responses for presentation to the class and giving those students some advance notice that they will be asked to explain what they have done;
- Sequencing those responses so that the reporting is cumulative; and
- Connecting the various strategies together.

Consolidating the Learning

So far, the lesson structure has facilitated the activation of cognition. The next phase is to provide opportunities for students to do what Dooley (2012) describes as *consolidating* the learning. This may involve posing a task similar in structure and complexity to the original challenge that helps to reinforce or extend the learning prompted by engagement with the original task.

Variation Theory offers a process that can guide the planning of these consolidating tasks. Kullberg, Runesson, and Mårtensson (2013), for example, described a study that used variation theory to plan lessons subsequent to an initial lesson on division of decimals. Their intention was that such task variations would prompt students to interpret the concepts in a different way from what they had seen previously. Kullberg et al. (2013) argued:

> In order to understand or see a phenomenon or a situation in a particular way one must discern all the critical aspects of the object in question simultaneously. Since an aspect *is noticeable only if it varies against a back-ground in invariance* (emphasis in original), the experience of variation is a necessary condition for learning something in a specific way. (p. 611)

In the application of Variation Theory to the creation of tasks intended to consolidate the learning prompted by the initial task, the intent is that some elements of the original task remain invariant, and other aspects vary so that the learner can focus on the concept and not be misled by over-generalisation from solutions to a single example. It is possible that this aspect is underemphasised in much commentary on student centred approaches.

In summary, the teachers' intentions include identifying the mathematical potential within a task; planning the elements of lessons that engage learners in creating their own solutions to problems including deliberately maintaining the challenge of the task; anticipating the need to differentiate the task for some students; effectively reviewing students' reporting on their activity on the task; and consolidating that learning through similar tasks thoughtfully varied. The overall project is continuing to explore all of these aspects.

The results below are intended to offer insights into the following research questions:

(a) To what extent is the proposed lesson structure manageable by teachers and to what extent does it support student engagement with the challenging tasks?

(b) How does the lesson structure connect student learning with subsequent teacher actions?

The Context of the EPMC Project and Processes of Data Collection

The data reported below were sought from teachers of students in Years 3/4 (ages 8 to 9) in schools serving communities across a variety of socio economic backgrounds. The project adopted a design research approach which "attempts to support arguments constructed around the results of active innovation and intervention in classrooms" (Kelly, 2003, p. 3). The key elements are an *intervention* by the researchers to propose (possibly) different pedagogies from those used normally, the approach is *iterative* in that subsequent interventions are based on previous ones, and the intent is that findings address issues of *practice*, in this case the structuring of lessons.

The first step in this iteration was a full-day professional learning session in which the teachers worked through a set of 10 tasks and lessons that focused on aspects of addition and subtraction. The 30 teachers were from 15 primary schools serving students from diverse socio economic backgrounds, mainly in metropolitan Melbourne. The teachers were a mix of age and experience, although skewed toward being more experienced. The teachers nearly all claimed to be confident that they know both the relevant mathematics and ways of teaching it.

The professional learning included teachers solving the task for themselves, discussing various solutions and considering pedagogies associated with each task and lesson. The importance of anticipating the student experience by exploring possible solutions and variations in advance was emphasised. Even though not part of this report, various strategies to elicit student motivation and persistence were suggested to the teachers.

After each lesson teachers completed a proforma, gathering data on the implementation of lessons using scaled and open responses. While there are advantages in observing lessons to examine the nature of implementation, such observations create interventions of their own and can make the data less representative of natural teaching. In the following, the data on the lessons are from self-report but it is stressed that the teachers were responding to a specific proforma immediately after having taught each lesson, offering readers confidence in the authenticity of the teachers' self-reports.

Teachers also completed additional summative surveys. The Likert-style items on the surveys were descriptive in form, and representative responses are reported below. The qualitative responses were read through and themes identified, especially where the responses aligned with aspects of the research questions.

The students completed an online instrument that included pre-assessments of content and some survey items. Similar questions were asked on a post-test. The main analysis of the test responses was through quantifying the types of student responses and comparing and considering changes in the profile of responses from pre-test to post-test.

The specifics of one lesson constitute the thrust of the data presented below. Even though this runs the risk of overgeneralising from a subset, the focus on this report is on the specifics of the structure of lessons and the details of one lesson elaborate the structure. Data on some other lessons are included for comparative purposes and to establish claims of wider applicability of the structure.

The lesson reported in detail below, titled *Making Both Sides Equal*, included the initial task, termed *learning task*, which was posed as follows:

> Work out some numbers that make both sides of these equations equal
>
> 898 + ? = 900 + ?
>
> 95 - ? = ? - 10
>
> Give a range of responses for each.

The main learning focus of this task is on equivalence; although there are aspects of pattern identification, partitioning and regrouping that might emerge. It is noted that equivalence is important mathematically. To emphasise this point, the 2013 examiners' report for the top level mathematics in Year 12 (17 year old students) (Victorian Curriculum and Assessment Authority, 2014) included the following statement:

> Equals signs should be placed between quantities that are equal—the working should not appear to be a number of disjointed statements. If there are logical inconsistencies in the student's working,

> full marks will not be awarded. For example, if an equals (sic) sign is placed between quantities that are not equal, full marks will not be awarded. (p. 1)

It is stressed that this is part of the first statement in the report from the examiners of the subject taken mainly by the very best students. Clearly it is important that students come to experience the notion of equivalence and there is no reason why students aged 8–9 years should not start to learn this.

It may not be obvious in what ways this task is challenging. Readers are invited to describe not only the relationship between the unknowns, especially in the subtraction example, but also the reasons that the relationship exists. It is such dimensions of the task that justify the categorisation of "challenging".

Through working on the task, it is hoped that, having found a number of solutions to the task, the patterns associated with creating the equal statements emerge. As with most of the other lessons, there is potential for multiple solutions. This has four benefits:

- It allows a low "floor" for the task in that all students can find at least one solution readily;
- There is an expectation that students will determine their own strategy for answering the questions and it is this opportunity for decision making that is engaging for the students;
- There is a high "ceiling" in which students who complete the learning task can seek to propose a generalisation; and
- Having found their own solution strategy, the openness means that students can make unique contributions to the class discussions.

The lesson documentation offered to the teachers also included a rationale for the lesson, the relevant extract from the *Australian Curriculum: Mathematics*, suggestions for a possible introduction to the lesson, and an indicative statement of the goals for student learning.

Enabling prompts (for students experiencing difficulty), which are intended to be posed separately, were suggested as follows:

> In your head, work out the number that would make this equation true:
> 9 + 6 = 10 + ?
> 9 – 5 = 7 - ?

Note that these use a similar structure to the learning task but with only one unknown and smaller numbers. If some students experience difficulty with the learning task, the teacher would present those students with one or both of these prompts after waiting an appropriate length of time. The intention is that, having completed the prompt(s), those students then proceed with the original learning task.

An extending prompt (for those who find solutions quickly) was suggested as follows:

> Describe the pattern that summarises all of your answers to the question.

One of the intentions of such prompts is to encourage students towards making a generalisation, in this case by finding a clear and concise way to describe the pattern of responses.

The "consolidating" task to follow up the initial learning was suggested as follows:

> What might be the missing numbers? Give at least 10 possibilities.
> 224 + ? = ? + 10

? – 10 = 100 – ?

Again, readers are invited to describe the relationship between the unknowns in the subtraction example, noting the ways that the relationship is both similar to and different from the earlier example. Readers are also reminded that there is no initial instruction other than clarifying relevant language.

Figure 2 presents the titles and learning tasks of a selection of four of the other nine lessons that are most similar in form and focus to *Making both sides equal*. Some data of teachers' responses to these lessons are presented below for comparison purposes. It is noted that the information to the teachers on the structuring of these lessons was similar to the lesson described above.

Lesson title	Learning Task
Addition Shortcuts	Work out the answer to 3 + 4 + 5 + 35 + 37 + 36 in your head. What advice would you give to a friend about how to work out answers to questions like these in their head?
Ways to Add in your Head	Work out how to add 298 + 35 in your head. What advice would you give to someone on how to work out answers to questions like this in their head?
Missing Number Subtraction	I did a subtraction question correctly for homework, but my printer ran out of ink. I remember it looked like 8 ☐ - 2 ☐ = ☐ 2 What might be the digits that did not print? (Give as many sets of answers as you can)
Two purchases	I bought a new pair of shoes and a new pair of sandals. The total cost was $87. I know that the shoes cost at least $50 more than the sandals. How much might the shoes cost?

Figure . The learning tasks of some other lessons in this iteration of the project.

Note that this final task is more complex that the others and requires different pieces of information to be processed simultaneously.

Results

The results are presented in two sections: teachers' reports of the implementation of different elements of the lesson structure; and students' responses to both a pre-test and post-test, including a follow-up discussion with teachers from a school with high improvement.

Reports of the Implementation of the Lesson Structure

The following presents the reactions of the teachers to the teaching of the lessons, seeking to answer the first of the research questions. It should be noted that the following represents a substantial data collection exercise in that around 30 teachers responded to a proforma immediately after teaching each of the 10 lessons (around 300 lessons in all).

Table 1 presents the profile of responses to general prompts about the *Making Both Sides Equal* lesson, rating the propositions from strongly disagree (SD) to strong agreement (SA). Note that the numbers of SD, disagree (D), and Unsure (U) responses were small so they have been aggregated.

Table 1
Teachers' Ratings of Aspects of the 'Making Both Sides Equal' Lesson Immediately After its Teaching (n = 30)

Prompts about the *Making both sides equal* lesson	SD, D, U	A	SA
The level of challenge was about right	4	17	9
I would use this lesson again even if I adapt it a little	1	13	16
Most students learned the main mathematical ideas	4	15	11
The contribution of the students to the discussion was good	4	14	12

The teachers endorsed these propositions (87% or more indicating agree or strongly agree). The most positive response was to the prompt about using the lesson again, which is a strong indication of the productivity and potential of the lesson, especially since the teachers had just taught it. The teachers were only slightly less positive about the students' learning. Note that it was not possible to differentiate teachers' responses based on background factors since those data were gathered anonymously. Overall the teachers gave very positive reactions to the lesson and the responses of the students.

Such positive responses were also evident in their responses to the other lessons. Table 2 presents summaries of responses to the comparison lessons. For ease of presentation, and recognising the potential of such analysis to be reductionist, responses of strongly disagree were allocated a score of 1, disagree 2, etc., and then those scores were averaged.

Table 2
Teachers' Ratings of Aspects of Other Lessons Immediately Their Teaching (n = 30)

	Addition Shortcuts	*Ways to Add in Head*	*Missing Number Subtraction*	*Two Purchases*
The level of challenge was about right	4.2	4.2	3.9	3.3
I would use this lesson again even if I adapt it a little	4.7	4.6	4.4	3.7
Most students learned the main mathematical ideas	4.0	4.0	3.7	3.4
The contribution of the students to the discussion was good	4.1	4.3	3.9	3.7

Overall these are very positive reports by the teachers to the lessons and the reactions of students, indicating that the responses to the *Making Both Sides Equal* lessons are representative of these other lessons as well. Even the responses to the more challenging *Two Purchases* lesson were very positive. The inference is that the teachers considered that the students engaged with the challenge, learned the mathematics and made productive contributions to discussions.

The teachers were also given the opportunity to provide written reactions to various open response prompts, some representative responses of which are presented below. In the post lesson proforma, some teachers commented on the engagement of the students during the *Making Both Sides Equal* lesson, especially with regard to the sharing of the learning:

> It was great to see every child have a go and once we came together and shared ideas the number of kids that were successful increased.
>
> I really enjoyed this lesson. I found it interesting and children were engaged.
>
> Children enjoyed the challenge and discussion was good.
>
> That they enjoyed having a go to equal both sides. … kids learnt from one another and were eager to go and fix their mistakes.
>
> Children enjoyed the lesson and were totally engaged. Although they found the concept bewildering at the start, they were still interested enough to persevere and complete the task, cross checking and evaluating as they went.

Many teachers also commented on the experience of the students with the concept of equivalence, such as:

> Great. It highlighted students' misconceptions of what = means
>
> I found this lesson valuable to show that the equal sign means the same as.

There were also teachers who reported on aspects of the challenge. For example:

> It was more difficult for students than I predicted. Again they generally used patterning well but did not always check it was accurate. The subtraction was more difficult.
>
> The difficulty they experienced with the concept—they tend to write e.g. 6+4=10+4 then want to do another problem. It was surprisingly hard to explain how each side needed to balance. After a while most got the idea and were then able to use pattern.

Of course, it does not matter that students find a task difficult—and indeed that is the intention—but it is critical that teachers are aware of student difficulties and take action to resolve them. This is addressed further below.

To explore ways that teachers implemented the various lesson elements, the post-lesson proforma sought an indication of the number of minutes teachers spent on each. Table 3 presents the mean in minutes for each element in each of the five comparison lessons.

Table 3
The Mean of the Duration in Minutes of the Lesson Elements (n = 28)

	Making Both Sides Equal	*Addition Shortcuts*	*Ways to Add in Head*	*Missing Number Subtraction*	*Two Purchases*
The introduction to the learning task	6.0	6.4	6.5	6.6	6.3
Students working on the learning task	16.0	15.2	12.1	15.8	17.0
Whole class review of the learning task	10.6	10.4	10.7	9.7	12.3
Introduction to the consolidating task	5.9	5.0	5.4	5.7	7.2
Students working on the consolidating task	16.9	14.7	17.4	15.9	16.7
Whole class review of the consolidating task	8.9	9.8	9.6	8.5	10.1

The most striking aspect of this is the similarity across lessons. Noting that this table presents summary data from 140 lessons, it seems that the lessons took around one hour (derived by adding up the mean times of the lesson elements), the teachers spent about 6 minutes introducing each of the tasks, the students spent around 15 minutes working on each of the tasks, and the teacher spent 10 further minutes on the whole class reviews of each task. Given the brief introductions, the extended time for students to work on a single task, even if differentiated, and time allocated to the review of their work, the inference is that teachers implemented the various lesson elements in the way that was recommended.

Another key aspect of the implementation of the lesson structure was the extent to which teachers reported using the prompts. As part of the post lesson proforma, the teachers noted the number of students who were given an enabling prompt, and the time they waited before giving the prompts. Table 4 presents of the mean and median of the number of students over the 28 lessons, the fewest and greatest number of students in any lesson given the prompts, and the average time that the teachers waited before giving out any prompts.

Table 4
The use of the enabling prompts over the 28 implementation of each of the lessons

Lesson title	Mean number of prompts given per lesson	Median number of prompts per lesson	Low number of prompts given in a single lesson	High number of prompts in a given lesson	Time until prompts given
Making Both Sides Equal	6.3	4	0	23	6.3
Addition Shortcuts	6.7	4	1	25	6.8
Finding Ways To Add In Your Head	6.2	4	0	20	6.6
Missing Number Subtraction	5.7	5	0	18	6.6
Two Purchases	10.9	10	1	23	7.0

To elaborate these data for the *Making Both Sides Equal* lesson, the teachers reported the enabling prompts were given to between two and six students, with the mode number being 4. There was one teacher who gave no prompts and three who gave prompts to more than 20 students. Nearly all teachers waited between five and 10 minutes before doing so. The intention is that all students first have opportunity to engage with the learning tasks and are only offered prompts after this opportunity. The data suggest that the prompts were implemented by teachers in ways compatible with the proposed lesson structure.

Table 5 presents similar data for the extending prompts.

Table 5
The use of the extending prompts over the 28 implementation of each of the lessons

Lesson title	Mean number of prompts given per lesson	Median number of prompts per lesson	Low number of prompts given in a single lesson	High number of prompts in a given lesson
Making Both Sides Equal	6.9	5	0	20
Addition Shortcuts	7.3	6	0	22
Finding Ways To Add In Your Head	7.9	6.5	0	22
Missing Number Subtraction	7.4	6.5	0	20
Two Purchases	3.3	1	0	20

In the *Making Both Sides Equal* lesson, only four teachers did not give out the extending prompt indicating that in most classes there were students for whom the learning task was not challenging. Most of the teachers gave the extending prompt to between one and ten students. There were three teachers who gave the extending prompt to more than 20 students. Noting the variability across the classes, overall this also suggests that such prompts were used judiciously.

Across all of the lessons, it seems that teachers made active and deliberate use of the prompts depending on the responses of the class. No teacher reported a negative response to the prompts which seem to be a useful device to differentiate learning opportunities while maintaining not only the challenge of the task but also a sense of the class as a learning community. This data in the table suggest that this aspect of the recommended lesson structure was implemented by the teachers.

Overall, this is compelling evidence, based on the teachers' reactions, that the ways they implemented the lesson structure aligned with the advice they were offered both as part of the professional learning and in the lesson documentation. This lesson structure is feasible and manageable and may have potential for transfer to other types of lessons as well.

Pre- and Post-Assessment of Student Learning

To gain a different indication of the implementation of the lesson, and to seek insights into whether participation in this and the other lessons improved the chances that students would answer associated assessment items correctly, students completed an online assessment before and after the set of 10 lessons. Three of the items sought responses to questions presented in a similar format to the *Making Both Sides Equal* tasks. Table 6 presents the overall results from the items. The prompt for each of the items was "What should be the "?" ?". The items were open response, which gives more confidence in the responses than had the items been multiple choice. Only responses from students in classes who completed both assessments are included. Even though the number range and placement of the unknown varies, it is arguable that the items are assessing similar mathematical knowledge to the *Making Both Sides Equal* lesson.

Table 6
Number (%) of Student Correct Responses to the Three Equivalence Items Pre and Post Implementation

	Pre-test n = 1050	Post-test n = 1080
100 + 56 = ? + 53	215 (20.3%)	497 (45.8%)
19 + 22 = 20 + ?	254 (25.0%)	487 (45.3%)
95 - ? = 75 - 10	180 (17.7%)	399 (37.2%)

The improvement is similar in all three items, with around 20 to 25% more of the group answering correctly after the intervention in comparison to before. In other words, about one quarter of the group improved overall. To put this another way, one third of those who could not respond correctly before the lessons could do so after the lessons.

Considering the responses to the three items together, in the post-test, 40% of the students got none of the three items correct, meaning 60% of the students got one or more correct, representing improvement compared to the individual items from the 32% of students who answered one or more correctly on the pre-test. That is, nearly half of the students who could not answer an item previously could now respond correctly at least once. Twenty five percent of the students answered all these items correctly on the post-test, an increase from 8% from the pre-test.

In short, a significant minority of students were better able to respond to the items after the lessons than they were before. Tests of proportions on the items are highly significant statistically ($p = .000$), but the issue is whether this constitutes a meaningful educational improvement. Indeed, it might have been expected that the improvement would be greater, given that the students had completed an apparently successful lesson specifically on the particular concept of equivalence and other lessons on related topics.

One possible interpretation is that these gains are impressive but that this type of learning takes longer than one lesson for many students and learning gains overall take time. This is exemplified by the modest gains on comparable items on the national numeracy assessment between Year 3 and Year 5, for example. Sullivan and Davidson (2014) noticed a comparable apparently limited gain on particular assessment items in a previous iteration of the project. They followed up with a delayed assessment using a pencil and paper format and also examined students' worksheets. From these, the new knowledge demonstrated on the assessments of the students was substantially greater than was revealed by the on line pre/post comparisons.

An interesting aspect of the results was that, in comparing results of school cohorts, it was noticed that there were quite wide variations in the extent of student improvement between the pre- and post-test. To explore this further, some teachers of a school who were particularly successful in terms of improvement in students' responses to the items, were asked whether they could explain the special results of their classes. These teachers' responses indicated that they:

- Allowed students the time to consolidate their learning;
- Specifically addressed the issue of student persistence;
- Worked through the tasks prior to the lessons to enhance their chances of anticipating student responses; and

- Used the same structure, incorporating each of the lesson elements, for each of the lessons.

The teachers also commented on ways they used the students' responses. For example, the summary phase after the learning task was described as follows:

> Discussion always ended with the learning task on the smart board and we allowed questions for clarification. Students worked independently. When it came to sharing, we made sure we had a range of strategies from least to most efficient which were all presented in different ways. All strategies were celebrated. Our main goal was to promote the most efficient strategies, but to try and show them in a range of different ways. Students learnt to articulate themselves clearly by listening to each other explain their thinking. It also validated their thinking, by listening to others.

Perhaps more critically, though, is the assessment information that they gathered:

> The learning tasks acted as Rich Assessment Tasks where we encouraged students to try their best because their strategies and attempts would help us to plan the follow up lessons. We used the results to plan the next sequence of lessons.

It is those words "we used the results to plan the next sequence of lessons" that may well be critical in consolidating the learning. While not directly connected to lesson structure, it does indicate that the cognition activated by the challenging task may need to be followed up by subsequent further challenges and explanations. Noting that none of the teachers in schools with less than average improvement were interviewed and so it is not clear how their responses might differ, it seems that the purposeful actions by the teachers of these classes with high gains have contributed to those gains.

Summary

The research reported above intended to explore the implementation of a particular lesson structure to exemplify the common advice on mathematics teaching, to offer teachers strategies for differentiating learning opportunities and to propose experiences to consolidate learning for the students.

The lesson that was the focus of this article was based on a challenging task intended to activate the thinking of students around the concept of equivalence and patterns. The lesson information offered to teachers included prompts for students who experienced difficulty and those who finished quickly and a consolidating task for all students. The teachers reported that the lesson was successful in terms of student learning and contribution to whole class discussion. The teachers reports of the time they spent on each of the lesson elements and the overall data suggests that they implemented the various elements as recommended. The reported frequency of use of the enabling and extending prompts indicated that this strategy was manageable and that teachers made active decisions on which and when students should be offered the prompts. There was satisfactory overall improvement in the students' responses presented. The comparative data from other lessons indicate that the responses were not idiosyncratic to the focus lesson but were comparable across the other lessons presented. It seems that the lesson structure is useful to teachers and may act as a guide in further teacher professional learning.

As indicated in the framework used to guide the research, it seems that teachers do make implementation decisions based on their knowledge about the mathematics (which was perhaps gained from the teacher professional learning day and other pre-lesson planning), about the pedagogy (which is mainly built into the lesson structure and associated advice), and about the students (which is partially gained from observing students closely while they interact with the tasks). Teachers do need to anticipate the constraints they might experience, such as negative student reactions and plan to address them. Teachers beliefs that students can solve problems for themselves are presumably reflected in the time allocated to the lesson elements and especially the time for students to work on the tasks.

The teachers of classes with impressive improvement between the pre- and post-test reported a series of actions that seemed productive. This suggests that having suitable tasks and lessons is necessary but not sufficient to ensure learning. Given the current interest in schools on improving students' responses to external assessments, further research on how to consolidate learning activated through challenging tasks would be useful.

References

Anthony, G., & Walshaw, M. (2009). *Effective pedagogy in mathematics*. *Educational Series—19.* Brussels: International Bureau of Education, Geneva.

Christiansen, B., & Walther, G. (1986). Task and activity. In B. Christiansen, A. G. Howson, & M. Otte (Eds.), *Perspectives on mathematics education* (pp. 243–307). Holland: Reidel.

Clark, C. M., & Peterson, P. L. (1986). Teachers' thought processes. In M. C. Wittrock (Ed.) *Handbook of research on teaching* (pp. 255–296). New York: Macmillan.

Desforges, C., & Cockburn, A. (1987). *Understanding the mathematics teacher: A study of practice in first schools*. London: The Palmer Press.

Dooley, T. (2012). Constructing and consolidating mathematical entities in the context of whole class discussion. In J. Dindyal, L. P. Cheng, & S.F. Ng (Eds.). *Mathematics education: Expanding horizons* (Proceedings of the 35th conference of the Mathematics Education Group of Australasia, Singapore. pp. 234–241). Singapore: MERGA.

Dweck, C. S. (2000). *Self-theories: Their role in motivation, personality, and development.* Philadelphia, PA: Psychology Press.

Hill, H., Ball, D., & Schilling, S. (2008). Unpacking pedagogical content knowledge: Conceptualising and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education, 39*(4), 372–400.

Inoue, N. (2010). Zen and the art of *neriage*: Facilitating consensus building in mathematics inquiry lessons through lesson study. *Journal of Mathematics Teacher Education*, Retrieved online from http://www.springerlink.com/content/g33022h2k1384461/.polt

Jackson, K., Garrison, A., Wilson, J., Gibbons, L., & Shahan, E. (2013). Exploring relationships between setting up complex tasks and opportunities to learn in concluding whole-class discussions in middle-grades mathematics instruction**.** *Journal for Research in Mathematics Education***,** *44*(4), 646–682.

Kelly, A. (2003). Research as design. *Educational Researcher, 32*(1), 3–4.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.

Kullberg, A., Runesson, U., & Mårtensson, P. (2013). The same task? Different learning possibilities. In C. Margolinas (Ed.), Task design in mathematics education. In *Proceedings of the International Commission on Mathematical Instruction (Study 22)* (pp. 609–616). Oxford, UK: ICMI.

Lappan, G., Fey, T., Fitzgerald, W. M., Friel, S., & Phillips, E. D. (2006). *Connected Mathematics 2: Implementing and teaching guide.* Boston, MA: Pearson, Prentice Hall.

Middleton, J. A. (1995). A study of intrinsic motivation in the mathematics classroom: A personal construct approach. *Journal for Research in Mathematics Education, 26*(3), 254–279.

National Council of Teachers of Mathematics. (2014). *Principles to action: Ensuring mathematics success for all.* Reston, VA: NCTM.

Organisation for Economic Co-operation and Development (2014). *Do students have the drive to succeed? PISA in Focus.* Paris: OECD Publishing.

Ruthven, K., Laborde, C., Leach, J., & Tiberghien, A. (2009). Design tools in didactical research: Instrumenting the epistemological and cognitive aspects of the design of teaching sequences. *Educational Researcher*, *38*(5), 329–342.

Smith, M. S., & Stein, M. K. (2011). *5 practices for orchestrating productive mathematical discussions.* Reston, VA: National Council of Teacher of Mathematics.

Stein, M. K., Grover, B. W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal, 33*(2), 455–488.

Sullivan, P., Askew, M., Cheeseman, J., Clarke, D., Mornane, A., Roche, A., & Walker, N. (2014). Supporting teachers in structuring mathematics lessons involving challenging tasks. *Journal of Mathematics Teacher Education*. DOI 10.1007/s10857-014-9279-2.

Sullivan, P., Clarke, D. M., & Clarke, B. (2013). *Teaching with tasks for effective mathematics learning.* New York: Springer.

Sullivan, P., & Davidson, A. (2014). The role of challenging mathematical tasks in creating opportunities for student reasoning. In J. Anderson, M. Cavanagh & A. Prescott (Eds.). *Curriculum in focus: Research guided practice*. (Proceedings of the 37th Annual Conference of the Mathematics Education Research Group of Australasia, Sydney, pp. 605–612). Sydney: MERGA.

Sullivan, P., Mousley, J., & Jorgensen, R. (2009). Tasks and pedagogies that facilitate mathematical problem solving. In B. Kaur, M. Kapur, & K. Berinderjeet, (Eds.), *Yearbook of the Association of Mathematics Educators* (pp. 17–42). London: AME and World Scientific Publishing.

Victorian Curriculum and Assessment Authority (2014). *2013 Examination report: Specialist Mathematics 1GA2.* Melbourne: VCAA.

Vygotsky, L. S. (1930–1934/1978). Mind in society: The development of higher psychological processes (M. Cole, V. John-Steiner, S. Scribner, & E. Souberman, Eds.). Cambridge, MA: Harvard University Press.

Zan, R., Brown, L., Evans, J., & Hannula, M. S. (2006). Affect in mathematics education: An introduction. *Educational Studies in Mathematics, 63*(2), 113–121.